Two-Dimensional Interleaving Schemes with Repetitions

Mario Blaum
IBM Research Division
650 Harry Road
San Jose, CA 95120, USA
`blaum@almaden.ibm.com`

Jehoshua Bruck *
California Institute of Technology
Mail Stop 136-93
Pasadena, CA 91125, USA
`bruck@paradise.caltech.edu`

Paddy Farrell
Electrical Engineering Laboratories
The University, Manchester
M13 9PL, England
`farrell@man.ac.uk`

Abstract

We present 2-dimensional interleaving schemes, with repetition, for correcting 2-dimensional bursts (or clusters) of errors, where a cluster of errors is characterized by its area. A recent application of correction of 2-dimensional clusters appeared in the context of holographic storage. Known interleaving schemes are based on arrays of integers with the property that every connected component of area t consists of distinct integers. Namely, they are based on the use of 1-error-correcting codes. We extend this concept by allowing repetitions within the arrays, hence, providing a trade-off between the error-correcting capability of the codes and the degree of the interleaving schemes.

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1 Introduction

We present 2-dimensional interleaving schemes, with repetitions, for correcting 2-dimensional bursts (or clusters) of errors, where a cluster of errors is characterized by its area. Known interleaving schemes are based on arrays of integers with the property that every connected component of area t consists of distinct integers. These arrays are called t-interleaved arrays. We extend the concept of t-interleaved arrays by allowing repetitions within the arrays. Namely, a t-interleaved array with repetition r is an array of integers with the property that in every connected component of area t, every integer is repeated at most r times. Next we formally define those concepts.

Definition 1.1 We say that an element (i, j) in a 2-dimensional array is *connected* to elements $(i+1, j)$, $(i-1, j)$, $(i, j+1)$ and $(i, j-1)$, provided those elements exist.

Definition 1.2 A *path* of length n from E_0 to E_n in a 2-dimensional array is a set of $n+1$ elements $\{E_i \,|\, 0 \leq i \leq n\}$ such that for every $0 \leq i < n$, element E_i is connected to element E_{i+1}.

Definition 1.3 We say that a set of t elements in a 2-dimensional array is a *cluster* of size t, if any two elements in the cluster belong in a path contained in the set.

The concept of a cluster of size t generalizes in two dimensions the concept of a burst of size t in one dimension. The same idea can be generalized to multiple dimensions (see [4]).

Example 1.1 The 1's in the array below constitute a cluster of size 7.

0	0	0	0	0	0	0
0	1	1	1	0	0	0
0	0	1	1	1	0	0
0	0	0	1	0	0	0

Definition 1.4 Let $t \geq 1$ and $r \geq 1$ be integers. Let $A(t, r)$ be a 2-dimensional array of integers, namely, the elements of the array are labeled by integers. We say that $A(t, r)$ is

t-interleaved with repetition r if every cluster of size t in $A(t,r)$ consists of integers that repeat at most r times. The *degree of interleaving* of the array is the number of distinct integers it contains.

Notice that, if the integers represent different codes (like in the one-dimensional case), then r-error-correcting codes distributed in a t-interleaved array with repetition r can correct any cluster of size up to t.

Example 1.2 The following is a $A(3,1)$ interleaved array, namely, it is 3-interleaved with repetition 1. Notice that the degree of interleaving 5:

$$A(3,1) = \begin{array}{|c|c|c|c|c|c|c|} \hline 0 & 1 & 2 & 3 & 4 & 0 & 1 \\ \hline 3 & 4 & 0 & 1 & 2 & 3 & 4 \\ \hline 1 & 2 & 3 & 4 & 0 & 1 & 2 \\ \hline 4 & 0 & 1 & 2 & 3 & 4 & 0 \\ \hline 2 & 3 & 4 & 0 & 1 & 2 & 3 \\ \hline \end{array}$$

Here we need only five 1-error correcting codes to correct any cluster of size 3.

Example 1.3 The following is $A(3,2)$, namely, it is 3-interleaved with repetition 2. Notice that the degree of interleaving 2:

$$A(3,2) = \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ \hline 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ \hline 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ \hline 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ \hline 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ \hline \end{array}$$

Here we need only two 2-error correcting codes to correct any cluster of size 3.

Our goal is to construct $A(t,r)$ arrays with minimal degree. Notice that in the one-dimensional case, the minimal degree of interleaving $\lceil t/r \rceil$ corresponds linearly to the size of the burst we want to correct. This is not the case in the 2-dimensional case, as we will see in the sequel. In [4] we presented optimal degree constructions of $A(t,1)$ arrays for arbitrary t. Here we focus on the case $r = 2$ and present lower bounds and constructions.

2 Lower Bounds

In this section we present lower bounds on the degree of interleaving of $A(t,r)$ arrays. We start by presenting the lower bound for the case $r = 2$, we then generalize it for arbitrary $r \geq 3$.

Theorem 2.1 Let $t \geq 2$. Let $A(t, 2)$ be a t-interleaved array with repetition 2. Then

1. If t is even, then the degree of interleaving of $A(t, 2)$ is at least $\frac{(t/2)(t/2+1)}{2}$.
2. If t is odd, then the degree of interleaving of $A(t, 2)$ is at least $\lceil \frac{(t+1)^2}{8} \rceil$.

Proof: The idea in the proof is to construct arrays of integers of size $\frac{(t/2)(t/2+1)}{2}$ in the case t even, and size $\lceil \frac{(t+1)^2}{4} \rceil$, in the case of t odd, and prove that if those arrays are part of $A(t, 2)$ then every integer in the array is repeated at most twice. We will prove the theorem for the case t even, the proof for the odd case follows by similar arguments.

We illustrate the idea by an example. Consider the following 2 by 3 array of labels called $B(4)$.

$$B(4) = \begin{array}{|c|c|c|} \hline a & b & c \\ \hline d & e & f \\ \hline \end{array}$$

Note that that every three labels in $B(4)$ are part of a cluster of size 4. This follows from the fact that for every three labels, two are in the same row. Namely, a cluster that contains this row and the third label is of size 4. For example, let a, c and e be the three labels then the cluster that contains those labels is a, b, c and e.

In general, for arbitrary t even, consider the array $B(t)$ to be a $t/2$ by $\lceil (t+1)/2 \rceil$ array. We need to prove that every three entries in $B(t)$ are part of a cluster of size t. The idea is to construct the cluster by picking the column (of size $t/2$) in $B(t)$ that contains the middle entry (horizontally) and connecting this column to the other two entries. The size of this cluster is at most $t/2 + t/2 = t$.

For example,

$$B(6) = \begin{array}{|c|c|c|c|} \hline a & b & c & d \\ \hline e & f & g & h \\ \hline i & j & k & l \\ \hline \end{array}$$

Let the three entries be a f and l. We construct the cluster by taking the column b, f and j and connecting it to a and l, resulting in the cluster $\{a, b, f, j, k, l\}$ of size 6.

We proved that every three entries in $B(t)$ are part of a cluster of size t. Hence, $A(t, 2)$ must be at least as large as $B(t)$ and the degree of interleaving of $A(t, 2)$ is at least $\lceil \frac{t(t+1)}{8} \rceil$. $\square$

By similar techniques we can prove the following general theorem.

Theorem 2.2 Let $t \geq 2$ and $r \leq t$. Let $A(t, r)$ be a t-interleaved array with repeatition r. Then the degree of interleaving of $A(t, r)$ is at least $\lceil \frac{t(t+1)}{2r^2} \rceil$.

Let us refine a little the lower bound given by Theorem 2.1 in the case $t = 6$. According to Theorem 2.1, the degree of interleaving of $A(6, 2)$ is at least 6. The next lemma shows that it is actually larger than 6.

Lemma 2.1 Let $A(6, 2)$ be a 6-interleaved array with repetition 2. Then the degree of interleaving of $A(6, 2)$ is at least 7.

Proof: According to Theorem 2.1, the degree of interleaving of $A(6, 2)$ is at least 6. Assume that it is exactly 6. Then, according to the proof of Theorem 2.1, every 3×4 rectangle and every 4×3 rectangle contain exactly 2 labels. Consider a label 0. Without loss of generality, we may assume that entry (0,0) in the plane is labeled with 0. For sure there is another 0 at Lee distance at most 5 from this 0, otherwise, we would have a 3×4 or 4×3 rectangle with only one 0. Graphically, we have to fill up

$$\begin{array}{c|c|c|c|c} \ddots & \vdots & \vdots & \vdots & \ddots \\ \hline \ldots & (-1,-1) & (-1,0) & (-1,1) & \ldots \\ \hline \ldots & (0,-1) & (0,0) & (0,1) & \ldots \\ \hline \ldots & (1,-1) & (1,0) & (1,1) & \ldots \\ \hline \ddots & \vdots & \vdots & \vdots & \ddots \end{array}$$

We will consider different cases for the second 0.

(0,1) So, we assume that the second 0 is in entry (0,1). Since every 3×4 or 4×3 rectangle contains exactly 2 0's, in particular, none of the entries (i, j), $1 \leq i \leq 3$, $-1 \leq j \leq 2$, can have a 0 label. In particular, this is a 3×4 rectangle with no zeros, a contradiction. The cases in which the second 0 is in (-1,0), (1,0) or (0,-1) are analogous, by symmetry.

(1,1) In this case, none of the entries $(i, j) \neq (1, 1)$, $-1 \leq i \leq 2$, $1 \leq j \leq 3$, can have a 0 label. But this is a 4×3 rectangle containing only one 0 label, a contradiction.

(1,2) In this case, none of the entries $(i, j) \neq (1, 2)$, $-1 \leq i \leq 2$, $1 \leq j \leq 3$, can have a 0 label. But this is a 4×3 rectangle containing only one 0 label, a contradiction.

(0,3) In this case, none of the entries $(i, j) \neq (0, 3)$, $-1 \leq i \leq 2$, $1 \leq j \leq 3$, can have a 0 label. This is a 4×3 rectangle containing only one 0 label, a contradiction.

(2,2) In this case, none of the entries $(i, j) \neq (0, 0), (-1, -1)$, $-1 \leq i \leq 2$, $-1 \leq j \leq 1$, can have a 0 label. Since this is a 4×3 rectangle, it contains exactly two 0 labels, therefore, (-1,-1) is labeled as 0. But this case is symmetrical to the one in which (1,1) is labeled as 0, which we saw that it leads to a contradiction.

(0,2) In this case, none of the entries $(i, j) \neq (0, 2), (2, 3)$, $-1 \leq i \leq 2$, $1 \leq j \leq 3$, can have a 0 label. Since this is a 4×3 rectangle, it contains exactly two 0 labels, therefore, (2,3) is labeled as 0. But the case in which (0,2) and (2,3) are labeled as 0 is symmetrical to the one in which (0,0) and (1,2) are labeled as 0, which we saw that leads to a contradiction.

(1,3) In this case, none of the entries $(i, j) \neq (1, 3)$, $-1 \leq i \leq 2$, $1 \leq j \leq 3$, can have a 0 label. But this is a 4×3 rectangle containing only one 0 label, a contradiction.

(2,3) In this case, none of the entries $(i, j) \neq (2, 3)$, $1 \leq i \leq 2$, $1 \leq j \leq 3$, can have a 0 label. Consider then the entries $(i, 4)$, $0 \leq i \leq 2$. Exactly one of these 3 entries has to be labeled with 0. If it is entry (0,4), this gives a contradiction by case (1,2), if it is entry (1,4), it gives a contradiction by case (1,1), and if it is entry (2,4), it gives a contradiction by case (0,1). $\square$

3 Constructions

In this section we present constructions of $A(2,t)$ interleaved arrays. First we describe an interleaving scheme that we call the *toroidal* interleaving scheme.

Construction 3.1 Consider a 2-dimensional array and an integer m. Label the coordinates of the array toroidally on m, i.e., the coordinates are given by (x,y), where x and y are taken modulo m. Let b be relatively prime with m. Then, for each a modulo m, the coordinates $(i, a+ib)$ (taken modulo m) are assigned the same number a.

Example 3.1 Assume that we have a 4×6 array. Taking $m=2$ and $b=1$, Construction 3.1 gives the following $A(3,2)$ interleaved array:

$A(3,2) =$

0	1	0	1	0	1
1	0	1	0	1	0
0	1	0	1	0	1
1	0	1	0	1	0

Similarly, if we consider a 5×10 array for $m=5$ and $b=3$, we obtain the following $A(5,2)$ interleaved array:

$A(5,2) =$

0	1	2	3	4	0	1	2	3	4
2	3	4	0	1	2	3	4	0	1
4	0	1	2	3	4	0	1	2	3
1	2	3	4	0	1	2	3	4	0
3	4	0	1	2	3	4	0	1	2

The reader can verify that the array above is 5-interleaved with repetition 2.

As we can see in Example 3.1, given an array labeled by Construction 3.1, in order to find if the array is t interleaved, it is enough to consider the $m\times m$ array obtained by the construction. The labeling of the whole array is obtained by tiling it with the $m\times m$ array.

Definition 3.1 The *Lee distance* between two elements in a torus is the length of the shortest path they belong to (for example, two adjacent elements are at Lee distance 1).

The minimum Lee distance of a set of elements is the minimum of the Lee distance between all the pairs of elements in the set.

The following lemma gives a method for finding t in Construction 3.1.

Theorem 3.1 Consider Construction 3.1 with parameters m and b. Let d be the minimum Lee distance in the $m \times m$ torus between two coordinates labeled with the same number. Then,

1. For d even, the construction provides an $A(t,2)$ interleaved array with $t = 3d/2$.
2. For d odd, the construction provides an $A(t,2)$ interleaved array with $t = (3d+1)/2$.

Proof: We will prove the theorem for the case d even. The odd case is similar. Notice that it is enough to consider the minimum Lee distance between those entries labeled with 0, i.e., between the coordinates (i, ib), $0 \leq i \leq m-1$.

Consider any three entries in the array labeled with a 0, call them a, b, and c. Assume that a, b and c are in a cluster of size $t = 3d/2$ and reach a contradiction. By the construction, without loss of generality, the Lee distance between a and b is d. Also, the Lee distance between a and c as well as between b and c is also d. Since a, b and c are in the same cluster, it must be of size at least $d+1$ (the path including a and b) plus $(d-1)/2$ (the size of the connection of a and b to c). Namely the cluster is of size at least $(3d+1)/2$ which is a contradiction. Namely, the array is a $A(t,2)$ interleaved. $\square$

The following theorem from [4] describes an optimal method for constructing arrays with a given minimum Lee distance d. Those arrays are in fact $A(d,1)$ interleaved arrays.

Theorem 3.2 Using construction 3.1 with parameters m and b provides constructions of arrays with minimum Lee distance d between equal labels. Where

1. For d an odd integer, $m = \frac{d^2+1}{2}$, and $b = d$.
2. For d an even integer, $m = \frac{d^2}{2}$, and $b = d-1$.

Corollary 3.1 Using construction 3.1 with parameters m and b provide constructions of $A(t, 2)$ interleaved arrays. Where,

1. For $t = (3d + 1)/2$, d an odd integer, $m = \frac{d^2+1}{2}$, and $b = d$. Resulting in a degree of interleaving of $\frac{2t^2-2t+5}{9}$.

2. For $t = 3d/2$, d an even integer, $m = \frac{d^2}{2}$, and $b = d - 1$. Resulting in a degree of interleaving of $\frac{2t^2}{9}$.

Example 3.2 Consider the case $d = 4$. According to Corollary 3.1 $m = 8$ and $b = 3$. Therefore, tiling an array with the following 8×8 array gives a $A(6, 2)$ interleaved array:

0	1	2	3	4	5	6	7
5	6	7	0	1	2	3	4
2	3	4	5	6	7	0	1
7	0	1	2	3	4	5	6
4	5	6	7	0	1	2	3
1	2	3	4	5	6	7	0
6	7	0	1	2	3	4	5
3	4	5	6	7	0	1	2

For $d = 5$, according to Corollary 3.1 $m = 13$ and $b = 5$. Therefore, tiling an array with the following 13×13 array gives a $A(8, 2)$ interleaved array:

0	1	2	3	4	5	6	7	8	9	10	11	12
8	9	10	11	12	0	1	2	3	4	5	6	7
3	4	5	6	7	8	9	10	11	12	0	1	2
11	12	0	1	2	3	4	5	6	7	8	9	10
6	7	8	9	10	11	12	0	1	2	3	4	5
1	2	3	4	5	6	7	8	9	10	11	12	0
9	10	11	12	0	1	2	3	4	5	6	7	8
4	5	6	7	8	9	10	11	12	0	1	2	3
12	0	1	2	3	4	5	6	7	8	9	10	11
7	8	9	10	11	12	0	1	2	3	4	5	6
2	3	4	5	6	7	8	9	10	11	12	0	1
10	11	12	0	1	2	3	4	5	6	7	8	9
5	6	7	8	9	10	11	12	0	1	2	3	4

We note that the constructions above are not optimal with respect to the degree of interleaving. In fact, we can improve the degree of interleaving obtained with construction 3.1 by using a different set of parameters. For example, we can construct $A(6,2)$ with degree 7, compared to degree 8 above, by using $m = 7$ and $b = 2$, as follows,

0	1	2	3	4	5	6
5	6	0	1	2	3	4
3	4	5	6	0	1	2
1	2	3	4	5	6	0
6	0	1	2	3	4	5
4	5	6	0	1	2	3
2	3	4	5	6	0	1

Using computer search we can find optimal sets of parameters for the torodial construction.

Next we present a general recursive construction. We will focus on the case where $t/4$ is an integer. The generalization is straightforward.

Construction 3.2 Let $t = 4k$, k an integer. Let $C_0(t)$ be the $\frac{t}{4} \times \frac{t}{4}$ array labeled by the integers $\{j \,:\, 0 \leq j \leq \frac{t^2}{16} - 1\}$, $C_1(t)$ be the $\frac{t}{4} \times \frac{t}{4}$ array labeled by the integers $\{j \,:\, \frac{t^2}{16} \leq$

$j \leq \frac{t^2}{8} - 1\}$ and $C_2(t)$ be the $\frac{t}{4} \times \frac{t}{4}$ array labeled by the integers $\{j : \frac{t^2}{8} \leq j \leq \frac{3t^2}{16} - 1\}$.

The $A(t, 2)$ interleaved array consists of the following tiling using the arrays $C_0(t)$, $C_1(t)$ and $C_2(t)$

$C_0(t)$	$C_1(t)$	$C_2(t)$
$C_2(t)$	$C_0(t)$	$C_1(t)$
$C_1(t)$	$C_2(t)$	$C_0(t)$

Example 3.3 Let $t = 4$. Then $A(4, 2)$ is obtained by tiling:

0	1	2
2	0	1
1	2	0

Let $t = 8$. Then

$$C_0(8) = \begin{array}{|c|c|} \hline 0 & 1 \\ \hline 2 & 3 \\ \hline \end{array}$$

$$C_1(8) = \begin{array}{|c|c|} \hline 4 & 5 \\ \hline 6 & 7 \\ \hline \end{array}$$

$$C_2(8) = \begin{array}{|c|c|} \hline 8 & 9 \\ \hline 10 & 11 \\ \hline \end{array}$$

And $A(8, 2)$ is obtained by tiling:

0	1	4	5	8	9
2	3	6	7	10	11
8	9	0	1	4	5
10	11	2	3	6	7
4	5	8	9	0	1
6	7	10	11	2	3

Theorem 3.3 For every $t = 4k$, the arrays $A_2(t)$ in Construction 3.2 are $A(t, 2)$ interleaved.

Proof: The proof follows by observing that a cluster connecting any three elements with the same label, say 0, must go through 4 blocks (each $t/4$ by $t/4$). Hence, it is of size at least t. $\square$

The following table summarizes the lower bounds and upper bounds on the degree of interleaving using the different methods.

t	Lower bound	Upper bound Theorem 3.2	Upper bound Optimization	Upper bound Theorem 3.3
3	2	2	2	
4	3		3	3
5	5	5	5	
6	6	8	7	
7	8		10	
8	10	13	12(1,5)	12
9	13	18	17(1,5)	
9	13	18	16(2,3)	
10	15		19(1,7)	
11	18	25	24(1,7)	
11			22(2,3)	
12	21	32	27(1,8)	27
13	25		33(1,9)	
14	28		37(1,10)	
15	32		44(1,10)	
16	36		48(1,11)	48
17	41		57(1,13)	
17			56(4,5)	
18	45		61(1,13)	
19	50		69(1,19)	
20	55		75(1,14)	75

Table 1: Lower and upper bounds on the degree of interleaving of $A(t, 2)$.

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